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Journal of Approximation Theory 138 (2006) 112–123
Journal of Approximation Theory
www.elsevier.com/locate/jat

Direct and inverse results in Hölder norms

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Received 19 November 2004; accepted 19 October 2005

Communicated by Vilmos Totik
Available online 15 December 2005

Abstract

We present a general approach to obtain direct and inverse results for approximation in Hölder norms. This approach is used to obtain a collection of new results related with estimates of the best polynomial approximation and with the approximation by linear operators of non-periodic functions in Hölder norms.

Keywords: Best approximation; Approximation by linear operators; Approximation in Hölder norm

1. Introduction

In last years, there have been some interest in studying the rate of convergence of different approximation processes in Hölder (Lipschitz) norms. The first one, due to A.I. Kalandiya [10], was motivated by applications in the theory of differential equations. Some improvements were obtained by N.I. Ioakimidis [9]. D. Elliot [8] gave other direct estimates. Later other papers were devoted to analyze approximation of periodic functions. For more historical comments on this subject we refer to [3].

The main subject of this paper is to present direct and converse results related with the best approximation and with approximation by linear operators of non-periodic functions in Hölder norms. This will be accomplished in the last section with the help of weighted moduli of smoothness associated to the so-called Ditzian–Totik moduli of smoothness.

In Section 2, we develop a general approach to show how to construct Hölder spaces $E_{\omega,\alpha}$ associated to a given modulus of smoothness ω on a Banach space E. Then, we introduce a modulus of smoothness $\theta_{\omega,\alpha}$ in this new space and characterize it in terms of an appropriated

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0021-9045/$ - see front matter
doi:10.1016/j.jat.2005.10.004

K-functional. In Section 3, we show how theorems concerned with approximation in the basic space E can be used to derive similar ones in the Hölder spaces $E_{\omega,\alpha}$. We remark that we are interested in applications of the abstract approach more than in a general theory in Banach spaces. Of course other results can be derived from our approach, we only include some important ones. This paper can be compared with [2] where approximation in Hölder norms is studied in the periodical case. We remark that the results of [2] can be deduced from the approach given here.

In what follows the letter E will denote a real Banach space which norm $\|\cdot\|_E$ and W a linear subspace of E with a seminorm $|\cdot|_W$.

2. Generalized Hölder spaces

There are different approaches to present generalized Hölder spaces. One of them assumes that we have in hand a certain modulus of smoothness. This last notion can be replaced by a K-functional when we are working with an abstract Banach space. In concrete examples one pass from a K-functional to a modulus of smoothness by means of a theorem which asserts that both notions are equivalent. There is a standard way to define what a K-functional is, but we cannot say the same for the notion of a modulus of smoothness of a given order. Thus, we begin this section by presenting a definition (convenient for our purposes) of a modulus of smoothness on a Banach space.

Definition 1. A modulus of smoothness on E is a function $\omega : E \times [0, +\infty) \to \mathbb{R}^+$ such that: (a) For each fixed $t \in (0, +\infty)$, the function $\omega(\cdot, t)$ is a seminorm on E and for all $f \in E$, $\omega(f, 0) = 0$; (b) For each fixed $f \in E$, the function $\omega(f, \cdot)$ is increasing on $[0, +\infty)$ and continuous at 0; (c) There exists a constant $C > 0$ such that for each $(f, t) \in E \times [0, +\infty)$, one has

$$\omega(f, t) \leqslant C \, \|f\| .$$

Given a real $r > 0$, we say that the modulus ω is of order r if $N(E, \omega, r) \neq Ker(\omega)$ and $N(E, \omega, s) = Ker(\omega)$ for all $s > r$, where

$$Ker(\omega) = \left\{ g \in E : \sup_{t \geqslant 0} \omega(g, t) = 0 \right\}$$

and

$$N(E, \omega, r) = \left\{ f \in E : \sup_{t>0} \frac{\omega(f, t)}{t^r} < \infty \right\} .$$

To each modulus of smoothness ω on E we associate some (generalized) Hölder spaces as follows.

Definition 2. Given a modulus of smoothness ω on E and a real $\alpha > 0$, we denote $\theta_{\omega,\alpha}(f, 0) = 0$,

$$\theta_{\omega,\alpha}(f, t) = \sup_{0<s\leqslant t} \frac{\omega(f, s)}{s^{\alpha}} \quad \text{and} \quad \|f\|_{\omega,\alpha} = \|f\|_E + \sup_{t>0} \theta_{\omega,\alpha}(f, t). \tag{1}$$

The Hölder space $E_{\omega,\alpha}$ is formed by those $f \in E$ such that $\|f\|_{\omega,\alpha} < \infty$ with the norm $\|f\|_{\omega,\alpha}$. Moreover we denote

$$E^0_{\omega,\alpha} = \left\{ f \in E_{\omega,\alpha} : \lim_{t\to 0} \theta_{\omega,\alpha}(f,t) = 0 \right\}.$$

Later we will prove that $\theta_{\omega,\alpha}$ is a modulus of smoothness of order $r-\alpha$ on $E^0_{\omega,\alpha}$ provided that ω is of order r. For the moment notice that $Ker(\theta_{\omega,\alpha}) = Ker(\omega)$. For completeness we recall the notion of K-functional.

Definition 3. If E and W are given as above, the K-functional K^W on E is defined for $f \in E$ and $t \geqslant 0$ by,

$$K^W(f,t) = \inf\{\|f-g\|_E + t|g|_W;\, g \in W\}.$$

If ω is a modulus of smoothness of order r on E, we say that ω and the K-functional K^W are equivalent if there are positive constants C_1, C_2 and t_0 such that for $f \in E$ and $t \in (0, t_0)$, we have

$$C_1\omega(f,t) \leqslant K^W(f,t^r) \leqslant C_2\omega(f,t). \tag{2}$$

Now we can state one of the main problems to be considered in this section. Given a linear space E, a real $r > 0$, $\alpha \in (0,r)$ and a modulus of smoothness ω of order r on E, characterize (1) in terms of a K-functional.

Since our approach will be used in concrete situations, it can be assumed that we have some additional information about ω. In many cases the proof of (2) is obtained as follows. It is shown that there exist positive constant C and t_0 such that for any $g \in W$ and $t \in (0, t_0]$,

$$\omega(g,t) \leqslant Ct^r|g|_W. \tag{3}$$

Moreover for each $t \in (0, t_0]$, there exists a function $L_t : E \to W$, such that for all $f \in E$,

$$\|f - L_t f\|_E \leqslant C\omega(f,t) \quad \text{and} \quad t^r|L_t f|_W \leqslant C\omega(f,t). \tag{4}$$

Notice that if (3) holds, then $W \subset E^0_{\omega,\alpha}$ ($\alpha \in (0,r)$). This fact will be used below.

In what follows we write $(E, W, L_t, \omega, r, \alpha, t_0)$ to assume that we have a Banach space E, a linear subspace W of E (with a seminorm $|\circ|_W \neq 0$), a modulus of smoothness ω of order r on E, and a family of functions $\{L_t\}$ such that conditions (3) and (4) hold and $\alpha \in (0,r)$.

Theorem 4. *If $(E, W, L_t, \omega, r, \alpha, t_0)$ is given as above, then $\theta_{\omega,\alpha}$ is a modulus of smoothness of order $r-\alpha$ on $E^0_{\omega,\alpha}$. Moreover if $g \in W$ and $t > 0$, then*

$$\theta_{\omega,\alpha}(f-g,t) \leqslant \|f-g\|_{\omega,\alpha} \quad \text{and} \quad \theta_{\omega,\alpha}(g,t) \leqslant Ct^{r-\alpha}|g|_W$$

(*where C is the constant given in* (4)) *and there exist positive constants D_1 and D_2 such that for $f \in E^0_{\omega,\alpha}$ and $t \in (0, t_0]$,*

$$D_1\theta_{\omega,\alpha}(f,t) \leqslant K_{\omega,\alpha}(f,t^{r-\alpha}) \leqslant D_2\theta_{\omega,\alpha}(f,t), \tag{5}$$

where

$$K_{\omega,\alpha}(f,t) = \inf\left\{\|f-g\|_{\omega,\alpha} + t|g|_W : g \in W\right\}.$$

Proof. If $f, g \in E_{\omega,\alpha}$, $a \in \mathbb{R}$ and $t \geqslant 0$, we have

$$\omega(f+g,t) \leqslant \omega(f,t)+\omega(g,t), \quad \omega(af,t) = \mid a \mid \omega(f,t).$$

Therefore $\theta_{\omega,\alpha}(f+g,t) \leqslant \theta_{\omega,\alpha}(f,t)+\theta_{\omega,\alpha}(g,t)$ and $\theta_{\omega,\alpha}(af,t) = \mid a \mid \theta_{\omega,\alpha}(f,t)$. On the other hand

$$\theta_{\omega,\alpha}(f-g,t) = \sup_{0<s\leqslant t} \frac{\omega(f-g,s)}{s^\alpha} \leqslant \sup_{s>0} \frac{\omega(f-g,s)}{s^\alpha} \leqslant \|f-g\|_{\omega,\alpha}.$$

Assume now that $g \in W$. Taking into account (3) we obtain

$$\theta_{\omega,\alpha}(g,t) = \sup_{0<s\leqslant t} \frac{\omega(g,s)}{s^\alpha} \leqslant C \sup_{0<s\leqslant t} s^{r-\alpha}|g|_W = Ct^{r-\alpha}|g|_W.$$

Fix $s > r-\alpha$. If $f \in N(E_{\omega,\alpha}, \theta_{\omega,\alpha}, s)$, then $\theta_{\omega,\alpha}(f,t) \leqslant C_f t^s$. Thus $\omega(f,t) \leqslant C_f t^{s+\alpha}$. This says that $f \in N(E,\omega,s) = Ker(\omega) = Ker(\theta_{\omega,\alpha})$. Hence $N(E_{\omega,\alpha}, \theta_{\omega,\alpha}, s) = Ker(\theta_{\omega,\alpha})$. Finally, if $f \in N(E,\omega,r) \setminus Ker(\omega)$, then $f \in N(E^0_{\omega,\alpha}, \theta_{\omega,\alpha}, r-\alpha) \setminus Ker(\theta_{\omega,\alpha})$. We have proved that $\theta_{\omega,\alpha}$ is a modulus of smoothness of order $r-\alpha$ on $E^0_{\omega,\alpha}$.

Fix $f \in E^0_{\omega,\alpha}$. For each $g \in W$,

$$\begin{aligned}\theta_{\omega,\alpha}(f,t) &\leqslant \theta_{\omega,\alpha}(f-g,t)+\theta_{\omega,\alpha}(g,t) \leqslant C_1 \left\{\|f-g\|_{\omega,\alpha}+\theta_\alpha(g,t)\right\} \\ &\leqslant C_1\left\{\|f-g\|_{\omega,\alpha}+t^{r-\alpha}|g|_W\right\}.\end{aligned}$$

Thus

$$\frac{1}{C_1}\theta_{\omega,\alpha}(f,t) \leqslant \inf\left\{\|f-g\|_{\omega,\alpha}+t^{r-\alpha}|g|_W : g \in W\right\} = K_{r,\alpha}(f,t^{r-\alpha}).$$

For the second inequality in (5) for each $t \in (0,t_0]$ we fix a function $L_t : E \to W$ which satisfies (4). For $s > t$ we obtain the estimates

$$\omega(f-L_tf,s) \leqslant C_2\|f-L_tf\|_E \leqslant C_3\omega(f,t) \leqslant C_3 s^\alpha \theta_{\omega,\alpha}(f,s).$$

Let us find a similar estimate for $s \leqslant t$. Recall that for $f \in E^0_{\omega,\alpha}$ and $t \in (0,t_0]$, $L_tf \in W$. Therefore for $s \in (0,t]$, we deduce from (3) and (4) that

$$\omega(L_tf,s) \leqslant C_4 s^r|L_tf|_W = C_4\left(\frac{s}{t}\right)^r t^r|L_tf|_W \leqslant C_5\left(\frac{s}{t}\right)^r \omega(f,t).$$

Thus, since K^W is a concave function and $s \leqslant t \leqslant t_0$

$$\omega(L_tf,s) \leqslant C_6 s^r \frac{K^W(f,t^r)}{t^r} \leqslant C_6 s^r \frac{K^W(f,s^r)}{s^r} \leqslant C_7\omega(f,s).$$

Now

$$\omega(f-L_tf,s) \leqslant \omega(L_tf,s)+\omega(f,s) \leqslant C_8\omega(f,s) \leqslant C_8 s^\alpha \theta_{\omega,\alpha}(f,s).$$

Therefore

$$\sup_{s>0} \frac{\omega(f-L_tt,s)}{s^\alpha} \leqslant C_8\theta_{\omega,\alpha}(f,t).$$

From the last inequality and (4) we infer that

$$\|f - L_t f\|_{\omega,\alpha} \leqslant C_9 \theta_{\omega,\alpha}(f,t)$$

and

$$t^{r-\alpha}|L_t t|_W \leqslant C_{10} \frac{\omega(f,t)}{t^\alpha} \leqslant C_{10} \theta_{\omega,\alpha}(f,t),$$

respectively. From this two last inequalities and the definition of a $K_{\omega,\alpha}$ we have

$$K_{\omega,\alpha}(f, t^{r-\alpha}) \leqslant \|f - L_t f\|_{\omega,\alpha} + t^{r-\alpha}|L_t f| \leqslant C_{11} \theta_{\omega,\alpha}(f,t). \qquad \square$$

3. Best approximation and linear approximation in Hölder spaces

In this section, we assume that there is a sequence $\{A_n\}_{n=0}^\infty$ of linear subspaces of E such that $A_n \subset A_{n+1}$, $\dim(A_n) = n$ and $\cup_{n=0}^\infty A_n$ is dense in E.

Recall that for $f \in E$ the best approximation of f by A_n is defined by

$$E_n(f) = dist(f, A_n) = \inf\{\|f - h\| : h \in A_n\}.$$

Theorem 5. *Let $(E, W, L_t, \omega, r, \alpha, t_0)$ be given as in the previous section and suppose that, for each n, $A_n \subset W$. For $f \in E^0_{\omega,\alpha}$ let $E_{n,\alpha}(f)$ be the best approximation of f (in $E_{\omega,\alpha}$) by A_n. If there exists a constant C_1 such that for each n, every $g \in W$ and each $h \in A_n$ one has*

$$E_{n,\alpha}(g) \leqslant C_1 \frac{1}{n^{r-\alpha}} |g|_W \quad \text{and} \quad |h|_W \leqslant C_1 n^{r-\alpha} \|h\|_E, \tag{6}$$

then there exist positive constants C_2 and C_3 such that for $f \in E^0_{\omega,\alpha}$ and each n one has

$$C_2 E_{n,\alpha}(f) \leqslant \theta_{\omega,\alpha}\left(f, \frac{1}{n}\right) \leqslant C_3 \frac{1}{n^{r-\alpha}} \sum_{k=1}^{n} k^{r-\alpha-1} E_{k,\alpha}(f). \tag{7}$$

Proof. From the main results in [4] we know that there exist positive constants C_4 and C_5 such that for every $f \in E^0_{w,\alpha}$ and every n,

$$C_4 E_{n,\alpha}(f) \leqslant K_{\omega,\alpha}\left(f, \frac{1}{n^{r-\alpha}}\right) \leqslant C_5 \frac{1}{n^{r-\alpha}} \sum_{k=1}^{n} k^{r-\alpha-1} E_{k,\alpha}(f).$$

Therefore the result follows from Eq. (5). $\square$

When a good approximation on E is obtained by means of an operator with a shape preserving property, then we can derive a direct-type result without using the first inequality in (6).

Theorem 6. *Let $(E, W, L_t, \omega, r, \alpha, t_0)$ be given as in the previous section and suppose that, for each n, $A_n \subset W$. If there exists a constant D and a sequence $\{H_n\}$ of functions, $H_n : E \to A_n$ such that, for each $f \in E$,*

$$\|f - H_n f\| \leqslant D\omega\left(f, \frac{1}{n}\right) \quad \text{and} \quad \omega(H_n f, t) \leqslant D\omega(f,t) \ \ (t > 0),$$

then for $h \in E^0_{\omega,\alpha}$ the first inequality in (7) holds.

Proof. If $f \in E^0_{\omega,\alpha}$, then for each n

$$\|f - H_n f\|_E \leqslant C_1 \omega\left(f, \frac{1}{n}\right) \leqslant C_1 \frac{1}{n^\alpha} \theta_{\omega,\alpha}\left(f, \frac{1}{n}\right).$$

On the other hand, for $t \geqslant 1/n$

$$\frac{\omega(f - H_n f, t)}{t^\alpha} \leqslant C_2 \frac{1}{t^\alpha} \|f - H_n f\|_E \leqslant C_3 \frac{1}{t^\alpha} \omega\left(f, \frac{1}{n}\right) \leqslant C_3 \theta_{\omega,\alpha}\left(f, \frac{1}{n}\right)$$

and, for $t \in (0, 1/n)$,

$$\frac{\omega(f - H_n f, t)}{t^\alpha} \leqslant \frac{\omega(f, t)}{t^\alpha} + \frac{\omega(H_n f, t)}{t^\alpha} \leqslant C_4 \theta_{\omega,\alpha}\left(f, \frac{1}{n}\right).$$

Therefore $E_{n,\alpha}(f) \leqslant \|f - H_n f\|_{\omega,\alpha} \leqslant D_4 \theta_{\omega,\alpha}(f, 1/n)$. □

Let us discuss some problems of approximation by linear operators in Hölder spaces. For the inverse estimate we need a result analogous to a lemma of Berens and Lorentz in [1]. Since the proof can be obtained with a modification of the one presented in [5, p. 312–313], we omit it.

Lemma 7. *If* $0 < \alpha < 2$, $a \in (0, 1)$ *and* ϕ *is an increasing positive function on* $[0, a]$ *with* $\phi(0) = 0$, *then for* $\beta \in (0, 2 - \alpha)$ *the inequalities* $\phi(a) \leqslant M_0 a^\beta$ *and* $\phi(x) \leqslant M_0 \left(y^\beta + (x/y)^{2-\alpha}\right)$ $(0 \leqslant x \leqslant y \leqslant a)$ *imply for some* $C = C(\alpha, \beta)$

$$\phi(x) \leqslant C M_0 x^\beta, \quad 0 \leqslant x \leqslant a.$$

Theorem 8. *Let* $(E, W, L_t, \omega, r, \alpha, t_0)$ *be given as in the previous section and suppose that, for each* n, $A_n \subset W$. *Let* $\{F_n\}$ *be a bounded sequence of linear operators for which there exist a constant* C *such that for each* $f \in E$, *every* $g \in W$ *and all* n, *one has* $F_n f \in A_n$ *and* $|F_n g|_W \leqslant C |g|_W$. *If for each* $f \in E$ *and every* n, *one has* $\|f - F_n f\| \leqslant D\omega(f, \psi(n))$, *where* $\{\psi(n)\}$ *is a decreasing sequence which converges to zero, then there exists a constant* D_1 *such that, for every* $h \in E^0_{\omega,\alpha}$, *and each* n

$$\|h - F_n h\|_{\omega,\alpha} \leqslant D_1 \theta_{\omega,\alpha}(h, \psi(n)). \tag{8}$$

Proof. To obtain (8) we only need to verify that $\sup_{t>0} t^{-\alpha} \omega(h - F_n h, t) \leqslant C_1 \theta_{\omega,\alpha}(h, \psi(n))$. If $t > \psi(n)$, then

$$\begin{aligned} \omega(h - F_n h, t) &\leqslant C_1 \|h - F_n h\|_E \leqslant C_2 \omega(f, \psi(n)) \\ &\leqslant C_2 \psi(n)^\alpha \theta_{\omega,\alpha}(f, \psi(n)) \leqslant C_3 t^\alpha \theta_{\omega,\alpha}(f, t). \end{aligned}$$

If $t \in (0, \psi(n)]$, then $\omega(h - F_n h, t) \leqslant C_4(\omega(h, t) + \omega(F_n h, t))$. Thus, it is sufficient to prove that $\omega(F_n h, t) \leqslant C_5 \omega(f, t)$. But

$$\begin{aligned} \omega(F_n h, t) &\leqslant C_6 \inf\left\{\|F_n h - g\|_E + t^r |g|_W \ : \ g \in W\right\} \\ &\leqslant C_6 \inf\left\{\|F_n h - L_n g\|_E + t^r |F_n g|_W \ : \ g \in W\right\} \\ &\leqslant C_7 \inf\left\{\|h - g\| + t^r |g|_W \ : \ g \in W\right\} \leqslant C_8 \omega(h, t). \qquad \square \end{aligned}$$

For approximation by linear operators different inverse results can be presented according to the classification given in [6]. We only consider some of them.

Theorem 9. *Assume the conditions given in theorem* 8 *with* $r = 2$. *If there exists a constant* C *such that for each* $f \in E$,

$$|F_n f|_W \leqslant Cn^2 \|f\|_E \quad \text{and} \quad |F_n g|_W \leqslant C\,|g|_W\,, \tag{9}$$

then there exists a constant D_1 *such that for each couple of positive integers n and k and* $f \in E^0_{\omega,\alpha}$ *one has*

$$\theta_{\omega,\alpha}\left(f, \frac{1}{n}\right) \leqslant D_1 \left\{\|f - F_k f\|_{\omega,\alpha} + \left(\frac{k}{n}\right)^{2-\alpha} \theta_{\omega,\alpha}\left(f, \frac{1}{k}\right)\right\}. \tag{10}$$

Moreover, if for $\beta \in (0, 2-\alpha)$ *and* $f \in E^0_{\omega,\alpha}$ *there exists a constant* C_f *such that,*

$$\|f - F_n f\|_{\omega,\alpha} \leqslant C_f \frac{1}{n^{\beta/2}} \tag{11}$$

for each positive integer n, then there exists a constant D_f *such that*

$$\theta_{\omega,\alpha}(f, t) \leqslant D_f t^{\beta}. \tag{12}$$

Proof. Fix $g \in W$ and integers n and k. From the definition of $K_{\omega,\alpha}$ and considering that $F_k f \in W \subset E_{\omega,\alpha}$ and the inequality (5) we obtain that there exists a positive constant C_1 such that

$$\begin{aligned} C_1\theta_{\omega,\alpha}\left(f, n^{-1}\right) &\leqslant K_{\omega,\alpha}\left(f, n^{\alpha-2}\right) \leqslant \|f - F_k f\|_{\omega,\alpha} + n^{\alpha-2}|F_k f|_W \\ &\leqslant \|f - F_k f\|_{\omega,\alpha} + n^{\alpha-2}\left(|F_k(f-g)|_W + |F_k g|_W\right) \\ &\leqslant \|f - F_k f\|_{\omega,\alpha} + n^{\alpha-2}k^2\left(\|f-g\|_E + k^{-2}|g|_W\right). \end{aligned}$$

We consider that $g \in W$ is arbitrary and use again (5), to infer that there exists a constant C_2 such that

$$\begin{aligned} C_1\theta_{\omega,\alpha}\left(f, n^{-1}\right) &\leqslant \|f - F_k f\|_{\omega,\alpha} + (k/n)^{2-\alpha} k^{\alpha} K_W\left(f, k^{-2}\right) \\ &\leqslant \|f - F_k\|_{\omega,\alpha} + C_2\,(k/n)^{2-\alpha}\,\theta_{\omega,\alpha}(f, 1/k)\,. \end{aligned}$$

This proves (10).

The estimate (12) is obtained from Lemma (7) and Eq. (10). □

4. Approximation of non-periodic functions

In this section, we realize the abstract approach presented above in the case of continuous or integrable functions defined on an interval of the real line. As before r is a fixed integer.

Here the letter I will always denote an interval of the real line and φ an admissible function in the sense of Ditzian–Totik (see [7, p. 8]). Recall that the function $\varphi(x) = \sqrt{x(1-x)}$, ($\sqrt{x}$, $\sqrt{x(1+x)}$) is admissible for the interval (0, 1) ((0, +∞)). For $p \in [1, +\infty)$, let $L_p(I)$ we denote the usual Lebesgue space with the norm $\|f\|_p = \left(\int_I |f(x)|^p\, dx\right)^{1/p}$. For $f \in L_p(I)$ and $t > 0$, the symmetric difference of order r, $\Delta_h^r f(x)$, is defined by

$$\Delta_h^r f(x) := \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} f\left(x + \left(\frac{r}{2} - j\right) h\right)$$

if $x \pm rh/2 \in I$ and it is considered as 0 in any other case.

For an admissible function φ the weighted (Ditzian–Totik) modulus of smoothness of order r is defined by

$$\omega_r^\varphi (f, t)_p := \sup_{h\in(0,t]} \|\Delta_{h\varphi}^r f\|_p.$$

Let $W_\varphi^{p,r}(I)$ denote the space of all $g \in L_p(I)$ such that, g is $r-1$ times differentiable, $g^{(r-1)}$ is absolutely continuous on each compact subinterval of I and $\|\varphi^r g^{(r)}\|_p < \infty$. In $W_\varphi^{p,r}(I)$ we consider the seminorm $|g|_{p,r} := \|\varphi^r g^{(r)}\|_p$. These notations are related to the ones considered in the previous section as follows $L_p(I) = E$, $\omega_r^\varphi(f, t)_p = \omega(f, t)$ and $W_\varphi^{p,r}(I) = W$ $(K_{r,\varphi}(f, t)_p = K^W(f, t))$.

It is easy to verify that $\omega_r^\varphi(f, t)_p$ is a modulus of smoothness of order r in the sense we have considered before. Thus for $\alpha \in (0, r)$ the Hölder space is well defined and we set $lip_{p,\alpha}^{\varphi,r}(I) = E_{\omega,\alpha}^0$, $\|\circ\|_{p,r,\alpha} = \|\circ\|_{\omega,\alpha}$, $\theta_{r,\alpha}^\varphi(f, t)_p = \theta_{\omega,\alpha}(f, t)$ and $K_{r,\varphi,\alpha}(f, t)_p = K_{\omega,\alpha}(f, t)$.

For the space $C(I)$ of bounded continuous functions we obtain similar definitions by changing the L_p norm by the sup norm. In this case, we use the last notations with $p = \infty$. In particular $L_\infty(I) = C(I)$.

From the proof of Theorem 2.1.1 in [7] we have

Theorem 10. *Fix* $1 \leqslant p \leqslant \infty$ *and an admissible function* φ *for I. There exist constants C and* t_0 *and, for each* $t \in (0, t_0]$ *a function* $L_t : L_p(I) \to W_\varphi^{p,r}(I)$ *such that for* $f \in L_p(I)$, $g \in W_\varphi^{p,r}(I)$ *and* $h > 0$,

$$\|\Delta_{h\varphi}^r g\|_p \leqslant C h^r \|\varphi^r g^{(r)}\|_p, \quad \|f - L_t f\|_p \leqslant C \omega_r^\varphi(f, t)_p \tag{13}$$

and

$$t^r \|\varphi^r (L_t f)^{(r)}\|_p \leqslant C \omega_r^\varphi(f, t)_p. \tag{14}$$

Moreover, there exist constant C_1 *and* C_2 *such that for* $t \in (0, t_0]$ *and* $f \in L_p(I)$

$$C_1 \omega_r^\varphi(f, t)_p \leqslant K_{r,\varphi}(f, t^r)_p \leqslant C_1 \omega_r^\varphi(f, t)_p. \tag{15}$$

Now we can state a similar theorem for spaces of Hölder functions. We remark that for the first inequality in (15) the restriction $t \leqslant t_0$ is not needed.

Theorem 11. *Fix* $\alpha \in (0, r)$. *Under the conditions of Theorem* 10 *there exist positive constants* D_1, D_2 *and* t_0 *such that for every* $f \in lip_{p,\alpha}^{\varphi,r}(I)$ *and* $t \in (0, t_0]$

$$D_1 \theta_{r,\alpha}^\varphi(f, t)_p \leqslant K_{r,\varphi,\alpha}(f, t^{r-\alpha})_p \leqslant D_2 \theta_{r,\alpha}^\varphi(f, t)_p. \tag{16}$$

Proof. We use Theorem 4. From (13) and (14) we know that conditions (3) and (4) hold. Then (16) follows from (5). □

Let Π_n denote the family of all algebraic polynomials of degree no greater than n. In order to use the results of Section 3, we set $\Pi_n = A_n$, $E_n(f)_p = E_n(f)$ and $E_{n,\alpha}(f)_p = E_{n,\alpha}(f)$. We first give a proof of the shape-preserving property needed in Theorem 6 and of the Bernstein-type inequality needed in Theorem 5. We remark that the result of Theorem 12 is seen to be known. Since it is important for us we include a proof.

Theorem 12. *Fix $1 \leqslant p \leqslant \infty$, a positive integer r and set $\varphi(x) = \sqrt{1-x^2}$ and $I = [-1, 1]$. For each n let $M_n : L_p(I) \to \Pi_n$ be a (non-linear) operator such that for each $f \in L_p(I)$, $\|f - M_n f\| = E_n(f)$. Then there exists a constant C such that for each $f \in L_p(I)$ and every $n > r$,*

$$\omega_r^{\varphi}(M_n f, t)_p \leqslant C\omega_r^{\varphi}(f, t)_p, t \in (0, 1/r].$$

Proof. From [7, p. 79, 84] we know that there exists a constant C_1 such that $(n > r)$

$$E_n(f)_p \leqslant C_1\omega_r^{\varphi}(f, 1/n)_p \quad \text{and} \quad \|\varphi^r (M_n f)^{(r)}\|_p \leqslant C_1 n^r \omega_r^{\varphi}\left(f, \frac{1}{n}\right)_p. \tag{17}$$

Recall that there exist constant D_1, D_2 and t_0 such that for $f \in L_p(I)$ and $t \in (0, t_0]$, Eq. (15) holds.

Fix a positive integer n, $f \in L_p(I)$ and $t > 0$. If $t > 1/n$, then

$$\begin{aligned}\omega_r^{\varphi}(M_n f, t)_p &\leqslant \omega_r^{\varphi}(f - M_n f, t)_p + \omega_r^{\varphi}(f, t)_p \\ &\leqslant C_2\|f - M_n f\|_p + \omega_r^{\varphi}(f, t)_p \leqslant C_3\omega_r^{\varphi}(f, t)_p.\end{aligned}$$

On the other hand, if $t \leqslant \min\{1/n, t_0\}$, then using (15) and (17) we obtain

$$\begin{aligned}\omega_r^{\varphi}(M_n f, t)_p &\leqslant C_4 K_{r,\varphi}(M_n f, t^r)_p \leqslant C_4 t^r \left\|\varphi^r (M_n f)^{(r)}\right\|_p \\ &\leqslant C_5 t^r n^r \omega_r^{\varphi}(f, 1/n)_p \leqslant C_6 t^r n^r K_{r,\varphi}(f, n^{-r})_p \\ &\leqslant C_6 K_{r,\varphi}(f, t^r)_p \leqslant C_7 \omega_r^{\varphi}(f, t)_p,\end{aligned}$$

where we have used the fact that $K_{r,\varphi}(f, t)_p$ is a concave function. From this we have the proof for the case $t \leqslant t_0$ $(t \leqslant 1/n)$. If $t > t_0$ $(t \leqslant 1/n)$, then using (15) we have

$$\begin{aligned}\omega_r^{\varphi}(M_n f, t)_p &\leqslant C_8 K_{r,\varphi}(M_n f, t^r)_p \leqslant C_8 \frac{t^r}{t_0^r} K_{r,\varphi}(M_n f, t_0^r)_p \\ &\leqslant C_8 \frac{1}{r^r t_0^r} K_{r,\varphi}(M_n f, t_0^r)_p \leqslant C_9 \omega_r^{\varphi}(f, t_0)_p. \qquad \square\end{aligned}$$

Theorem 13. *Set $I = [-1, 1]$ and $\varphi(x) = \sqrt{1-x^2}$. Fix $0 \leqslant p \leqslant +\infty$, a positive integer r and $\alpha \in (0, r)$. There exists a constant C such that, for any positive integer n and every $P \in \Pi_n$*

$$\|\varphi^r P^{(r)}\|_p \leqslant C n^{r-\alpha} \|P\|_{p,\alpha}.$$

Proof. We present a proof for $p < \infty$. For $p = \infty$ similar arguments can be used. If P is a polynomial of degree n, then $dist(P, \Pi_n) = 0$. Thus from the second inequality in (17) it follows that

$$\|\varphi^r P^{(r)}\|_p \leqslant C_1 n^r \omega_\varphi^r \left(P, \frac{1}{n}\right)_p \leqslant C_1 n^{r-\alpha} \theta_{\omega,\alpha}\left(P, \frac{1}{n}\right)_p \leqslant C_2 n^{r-\alpha} \|P\|_{p,\alpha},$$

where we have considered Theorem 4. □

Theorem 14. *Set $I = [-1, 1]$ and $\varphi(x) = \sqrt{1-x^2}$. Fix $0 \leqslant p \leqslant +\infty$, a positive integer r and $\alpha \in (0, r)$. Then there exist positive constants C_1 and C_2, such that, for every $f \in lip_{p,\alpha}^{\varphi,r}(I)$ and all $n > r$*

$$C_1 E_{n,\alpha}(f)_p \leqslant C_1 \theta_{r,\alpha}\left(f, \frac{1}{n}\right)_p \leqslant C_2 \frac{1}{n^{r-\alpha}} \sum_{k=1}^{n} k^{r-\alpha-1} E_{k,\alpha}(f)_p.$$

Proof. The first inequality follows from Theorem 6, Eq. (17) and Theorem 12. The inverse inequality follows from Theorem 5, since we have verified the Bernstein-type inequality in Theorem 13. □

Recall that for a real function f on [0, 1] the Bernstein polynomial is given by

$$B_n(f, x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}.$$

For these operators we consider the weight function $\varphi(x) = \sqrt{x(1-x)}$ and set $E = C[0, 1]$ and $F = lip_{p,\alpha}^{\varphi,2}[0, 1]_\infty$.

For $f \in L_1[0, 1]$ and a positive integer n the Kantarovich polynomial are defined by

$$K_n(f, x) = (n+1) \sum_{k=0}^{n} \left(\int_{k/(n+1)}^{(k+1)/(n+1)} f(s)\, ds \right) \binom{n}{k} x^k (1-x)^{n-k}.$$

For these operator we consider the weight function $\varphi(x) = \sqrt{x(1-x)}$ and set $E = L_p[0, 1]$ and $F = lip_{p,\alpha}^{\varphi,2}[0, 1]_p$.

For $f \in C_\infty[0, +\infty)$ and a positive integer n, the Szasz–Mirakyan operator is given by

$$S_n(f, x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right).$$

For these operators we consider the weight function $\varphi(x) = \sqrt{x}$ and set $E = C_\infty[0, \infty)$ and $F = lip_{p,\alpha}^{\varphi,2}[0, \infty)_\infty$.

For $f \in L_p[0, +\infty)$ the operators of Szasz–Kantarovich are defined as

$$S_n^*(f, x) = e^{-nx} \sum_{k=0}^{\infty} \left(\int_{k/(n+1)}^{(k+1)/(n+1)} f(s)\, ds \right) \frac{(nx)^k}{k!}.$$

In this case we consider the weight $\varphi(x) = \sqrt{x}$ and the spaces $E = L_p[0, \infty)$ and $F = lip_{p,\alpha}^{\varphi,2}[0, \infty)_p$.

For $f \in C_\infty[0,+\infty)$, the Baskakov operators are defined by

$$V_n(f,x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right)\binom{n+k-1}{k} x^k (1+x)^{-n-k}.$$

In this case we consider the weight $\varphi(x) = \sqrt{x(1+x)}$ and set $E = C_p[0,\infty)$ and $F = lip_{p,\alpha}^{\varphi,2}[0,\infty)_\infty$.

The Baskakov–Kantarovich polynomials are defined analogously. In this case we consider the weight $\varphi(x) = \sqrt{x(1+x)}$ and set $E = L_p[0,\infty)$ and $F = lip_{p,\alpha}^{\varphi,2}[0,\infty)_p$.

Theorem 15. *Let $\{F_n\}$ be the sequence of Bernstein (Kantarovich, Szasz–Mirakyan, Szasz–Kantarovich, Baskakov) operators with the weight function φ and the associated space E and F be given as above where $\alpha \in (0,2)$.*

(i) *There exist a constant C such that, for $f \in F$ and each positive integer n*

$$\|f - F_n(f)\|_{w,\alpha} \leqslant C\theta_{2,\alpha}^{\varphi}\left(f, \frac{1}{\sqrt{n}}\right)_p.$$

(ii) *For $k \leqslant n$ one has*

$$\theta_{r,\alpha}^{\varphi}\left(f, \frac{1}{n}\right) \leqslant D_1\left\{\|f - F_k f\|_{p,2,\alpha} + \left(\frac{k}{n}\right)^{2-\alpha} \theta_{2,\alpha}^{\varphi}\left(f, \frac{1}{k}\right)\right\}.$$

(iii) *Fix $\beta \in (0, 2-\alpha)$ and $f \in F$. There exists a constant C_f such that, for all n,*

$$\|f - F_n f\|_{p,2,\alpha} \leqslant C_f \frac{1}{n^{\beta/2}}$$

if and only if there exists a constant D_f such that

$$\theta_{2,\alpha}^{\varphi}(f,t) \leqslant D_f t^\beta.$$

Proof. It follows from Theorem 9.3.2 in [7, p. 117] that,

$$\|f - F_n(f)\|_p \leqslant C\left\{\frac{1}{n}\|f\|_p + \omega_2^{\varphi}\left(f, \frac{1}{\sqrt{n}}\right)_p\right\}.$$

On the other hand, there exists a constant D such that, for any $g \in W$,

$$\|\varphi^2 F_n^{(2)} g\|_p \leqslant D_2 \|\varphi^2 g^{(2)}\|_p$$

(see (9.3.7) in [7, p. 118]). Then the result follows from Theorem 8.

(ii) For the inverse result we only need to verify condition (9), that is the Bernstein type inequality $\|\varphi^2 L_n^{(2)} f\|_p \leqslant C n^2 \|f\|_p$. But this last inequality is known (see Eq. (9.3.5) in [7, p. 118]).

(iii) It is a consequence of (i) and (ii). □

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